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Alternating Sign Matrices and Descending Plane Partitions

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An *alternating sign matrix* is a square matrix such that (i) all entries are 1, −1, or 0, (ii) every row and column has sum 1, and (iii) in every row and column the nonzero entries alternate in sign. Striking numerical evidence of a connection between these matrices and the descending plane partitions introduced by Andrews (*Invent. Math.* **53** (1979), 193–225) have been discovered, but attempts to prove the existence of such a connection have been unsuccessful. This evidence, however, did suggest a method of proving the Andrews conjecture on descending plane partitions, which in turn suggested a method of proving the Macdonald conjecture on cyclically symmetric plane partitions (*Invent. Math.* **66** (1982), 73–87). In this paper is a discussion of alternating sign matrices and descending plane partitions, and several conjectures and theorems about them are presented.

1. DEFINITIONS

We begin with a definition.

DEFINITION 1. An *alternating sign matrix* is a square matrix which satisfies:

(i) all entries are 1, −1, or 0,

(ii) every row and column has sum 1,

(iii) in every row and column the nonzero entries alternate in sign.

All permutation matrices are alternating sign matrices. For 1×1 and 2×2 matrices these are the only alternating sign matrices. There are exactly seven alternating sign 3×3 matrices, six permutation matrices and the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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There are 42 alternating sign 4×4 matrices, 24 permutation matrices, 16 with one -1, and 2 with two -1's.

Let A_n be the number of $n \times n$ alternating sign matrices, and set

$$D(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

We have observed that $A_n = D(n)$ for $n \leqslant 10$. Thus, we make the following conjecture:

Conjecture 1. $A_n = D(n)$ for all n.

We shall give additional numerical results that support this conjecture in the next section.

DEFINITION 2. A *shifted plane partition* is an array $\pi = (a_{ij})$ of positive integers, defined only for $j \geqslant i$, that has nonincreasing rows and columns, and that can be written in the form

$$\begin{array}{ccccccc} a_{11} & a_{12} & a_{13} & \cdots & & & a_{1\mu_1} \\ & a_{22} & a_{23} & \cdots & & a_{2\mu_2} & \\ & & & \cdots & & & \\ & & & \cdots & & & \\ & & a_{rr} & \cdots & a_{r\mu_r}, & & \end{array}$$

where:

(i) $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$,

(ii) $a_{ij} \geqslant a_{i,j+1}$ whenever both sides are defined,

(iii) $a_{ij} \geqslant a_{i+1,j}$ whenever both sides are defined.

For technical reasons the empty array $\varnothing$ is taken to be a shifted plane partition. The *parts* of a shifted plane partition are the numbers that appear in the array. The *weight* of a shifted plane partition is the sum of its parts.

With such a partition we associate its "Ferrers graph" $F(\pi)$, which is the set of all integer points (i, j, k) for which a_{ij} is defined and $1 \leqslant k \leqslant a_{ij}$.

DEFINITION 3. A *strict shifted plane partition* is a shifted plane partition in which condition (iii) is replaced by

(iv) $a_{ij} > a_{i+1,j}$ whenever both sides are defined.

Thus the term strict means that the columns are strictly decreasing.

DEFINITION 4. A *descending plane partition* is a strict shifted plane partition such that:

(v) $a_{ii} > \lambda_i$, $1 \leqslant i \leqslant r$,

(vi) $a_{ii} \leqslant \lambda_{i-1}$, $1 < i \leqslant r$,

where $\lambda_i = \mu_i - i + 1$ is the number of elements in the ith row.

Conditions (v) and (vi) imply condition (i).

R. P. Stanley observed that $D(n)$ is the number of descending plane partitions with no parts exceeding n, a result that has been proved by Andrews [1]. A simpler proof can be found in [2]. It was this observation that gave rise to our interest in descending plane partitions and eventually to many of the results in this paper.

For example there are exactly seven descending plane partitions with parts not exceeding 3. These are

$$\varnothing, \quad 2, \quad 3, \quad 3\ 1, \quad 3\ 2, \quad 3\ 3, \quad \begin{matrix} 3 & 3, \\ & 2 \end{matrix}$$

which is in agreement with the number of 3×3 alternating sign matrices.

2. Refinements of the Main Conjecture

In this section we discuss two more conjectures, each of which implies our main conjecture. Conjecture 2 is a simple refinement of Conjecture 1, which involves alternating sign matrices only. It was Conjecture 2 that actually came first and led us directly to Conjecture 1. Conjecture 3 suggests the existence of a "natural" one-to-one correspondence between $n \times n$ alternating sign matrices and descending plane partitions with no parts exceeding n. Since we already know that the number of such descending plane partitions is $D(n)$, this one-to-one correspondence would establish Conjecture 1.

We note that an alternating sign matrix has a single nonzero element in the top row, which must be a 1. We classify the alternating sign matrices by where this 1 occurs. Thus, we let $A(n, k)$ denote the number of $n \times n$ alternating sign matrices with the 1 in the top row occurring in the kth position. It is sometimes convenient to write the $A(n, k)$ as a sort of Pascal's triangle of counts,

$$\begin{matrix}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 2 & & 3 & & 2 & & \\
 & 7 & & 14 & & 14 & & 7 & \\
42 & & 105 & & 135 & & 105 & & 42 \\
 & & \vdots & & \vdots & & \vdots, & &
\end{matrix}$$

where the nth row contains $A(n, 1)$, $A(n, 2), \ldots, A(n, n)$. For example, there are exactly 14 alternating sign 4×4 matrices with a 1 in the second position of the top row.

We now consider the ratios between horizontally adjacent entries in each row. If we insert these ratios, scaled as shown, between the corresponding elements in our triangle, then we get

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 2:2 & 1 & & & \\
 & & 2 & 2:3 & 3 & 3:2 & 2 & & \\
 & 7 & 2:4 & 14 & 5:5 & 14 & 4:2 & 7 & \\
42 & 2:5 & 105 & 7:9 & 135 & 9:7 & 105 & 5:2 & 42 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We notice that the left and right sides of these ratios separately satisfy the recursion of Pascal's triangle. In other words, the sum of two consecutive left sides of ratios in any row is equal to the left side of the appropriate ratio in the next row, and similarly for the right sides of the ratios. This observation and the obvious boundary conditions are equivalent to

Conjecture 2. If $0 < k < n$, then

$$A(n, k+1)/A(n, k) = (n-k)(n+k-1)/k(2n-k-1).$$

This conjecture has been verified numerically for $n \leqslant 10$. We will next show that it implies Conjecture 1.

First, we note that

$$A_n = \sum_{k=1}^{n} A(n, k). \tag{1}$$

Secondly, if an $n \times n$ alternating sign matrix has a 1 in its upper left-hand corner, then the other elements of the first row and the first column are all zero, and furthermore the other rows and columns form an $(n-1) \times (n-1)$ alternating sign matrix. Thus, we have

$$A(n, 1) = A_{n-1}. \tag{2}$$

Now, suppose that Conjecture 2 holds. Then we have

$$\begin{aligned}
\frac{A(n, r)}{A(n, 1)} &= \prod_{k=1}^{r-1} \frac{(n-k)(n+k-1)}{k(2n-k-1)} \\
&= \binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}\Big/\binom{2n-2}{n-1}.
\end{aligned}$$

Summing this over r, using (1) and (2), and replacing n by $n+1$, we obtain

$$\begin{aligned}\frac{A_{n+1}}{A_n} &= \sum_{r=1}^{n+1} \frac{A(n+1,r)}{A(n+1,1)} \\ &= \sum_{r=1}^{n+1} \binom{n+r-1}{n}\binom{2n-r+1}{n}\Big/\binom{2n}{n} \\ &= \sum_{r=0}^{n} \binom{n+r}{n}\binom{2n-r}{n}\Big/\binom{2n}{n} = \binom{3n+1}{n}\Big/\binom{2n}{n}.\end{aligned}$$

On the other hand, we have

$$\frac{D(n+1)}{D(n)} = \binom{3n+1}{n}\Big/\binom{2n}{n},$$

and $A_1 = 1 = D(1)$. It follows that if Conjecture 2 holds, then $A_n = D(n)$ for all n, which is Conjecture 1.

It is easy to show that Conjecture 2 holds for $k=1$ and hence, for $k=n-1$. So far we have been unable to handle even $k=2$.

We now discuss the evidence in favor of a "natural" one-to-one correspondence between alternating sign matrices and descending plane partitions.

A descending plane partition with no parts exceeding n can have at most $n-1$ parts that are equal to n. Let $D(n,k)$ denote the number of descending plane partitions with no parts exceeding n, that contain exactly $k-1$ parts that are equal to n. We have verified that $A(n,k) = D(n,k)$ for $n \leqslant 10$ and indeed we conjecture that this is true for all n. This is actually a consequence of our next conjecture, so we shall not list it separately.

Suppose that $M = (a_{ij})$ is an alternating sign matrix. We define the number of inversions in M to be $\sum a_{ij}a_{kl}$, where the summation is over all i, j, k, l such that $i < k$ and $j > l$. It is easy to see that this generalizes the usual notion of inversions for permutation matrices. It can be shown that the number of inversions is a nonnegative integer. Our numerical evidence strongly suggests that the number of inversions of an alternating sign matrix corresponds to the number of parts in a descending plane partition.

Another number associated with an alternating sign matrix is the number of -1's in it. We shall say that a part a_{ij} of a descending plane partition is *special* if $a_{ij} \leqslant j-i$, i.e., if a_{ij} does not exceed the number of entries to its left in its row. The -1's in an alternating sign matrix appear to correspond to the special elements of a descending plane partition. Indeed, we make

Conjecture 3. Suppose that n, k, m, p are nonnegative integers, $1 \leqslant k \leqslant n$. Let $\mathscr{A}(n,k,m,p)$ be the set of alternating sign matrices such that

(i) the size of the matrix is $n \times n$,

(ii) the 1 in the top row occurs in position k,

(iii) the number of -1's in the matrix is m,

(iv) the number of inversions in the matrix is p.

On the other hand, let $\mathscr{D}(n, k, m, p)$ be the set of descending plane partitions such that

(I) no parts exceed n,

(II) there are exactly $k-1$ parts equal to n,

(III) there are exactly m special parts,

(IV) there are a total of p parts.

Then $\mathscr{A}(n, k, m, p)$ and $\mathscr{D}(n, k, m, p)$ have the same cardinality.

Conjecture 3 has been verified numerically for all $n \leqslant 7$. It has been proved for the case $m=0$, as well as for a number of additional special cases.

As an example consider the case $n=5$, $k=3$, $m=1$, $p=4$. Here, $\mathscr{A}(5, 3, 1, 4)$ consists of the following 10 alternating sign matrices:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, $\mathscr{D}(5, 3, 1, 4)$ consists of the following 10 descending plane partitions:

$$5\ 5\ 3\ 1, \quad 5\ 5\ 3\ 2, \quad 5\ 5\ 3\ 3, \quad 5\ 5\ 4\ 1, \quad 5\ 5\ 4\ 2, \quad 5\ 5\ 4\ 3,$$

$$\begin{array}{ccc} 5 & 5 & 1, \\ & 2 & \end{array} \quad \begin{array}{ccc} 5 & 5 & 1, \\ & 3 & \end{array} \quad \begin{array}{ccc} 5 & 5 & 2, \\ & 2 & \end{array} \quad \begin{array}{ccc} 5 & 5 & 2. \\ & 3 & \end{array}$$

We now set $D(n, k, m, p) = |\mathscr{D}(n, k, m, p)|$, the number of descending plane partitions in $\mathscr{D}(n, k, m, p)$. We have obtained the following generating function identity:

$$\sum_{k,m,p} D(n, k, m, p)\, w^{k-1} x^m y^p = \det(G + I_{n-1}),$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix and G is the $(n-1) \times (n-1)$ matrix given by

$$G_{ij} = \sum_{s=1}^{j} \sum_{t=0}^{s-1} \binom{i}{s-1-t} \binom{s-1}{t} x^t y^s$$

for $1 \leqslant i \leqslant n-2$, $1 \leqslant j \leqslant n-1$; and

$$G_{n-1,j} = \sum_{k=1}^{j} \sum_{s=k}^{j} \sum_{t=0}^{s-k} \binom{n-k-1}{s-t-k} \binom{s-1}{t} w^k x^t y^s$$

for $1 \leqslant j \leqslant n-1$.

We intend to derive this identity in a subsequent paper. It is analogous to a generating function identity proved by Andrews [1].

3. The Partial Ordering of Descending Plane Partitions

Let $\mathscr{D}_n$ be the set of all descending plane partitions with no parts exceeding n. In this section, we define a partial ordering for this set of descending plane partitions. Thus, we can regard $\mathscr{D}_n$ as a partially ordered set, and we shall show that this partially ordered set has a unique antiautomorphism. We have adopted the working hypothesis that this antiautomorphism corresponds to the reversal of the order of the columns of alternating sign matrices. This hypothesis has led to several of the conjectures in this paper, most notably Conjecture 3 and its symmetric variation Conjecture 3S, that appears at the end of this section.

Let $\pi = (a_{ij})$ and $\pi_1 = (b_{ij})$ be descending plane partitions. We shall write $\pi \geqslant \pi_1$ if a_{ij} is defined and $a_{ij} \geqslant b_{ij}$ for all i and j for which b_{ij} is defined. This defines a partial ordering among the descending plane partitions. It can be readily verified that this ordering gives us a lattice of descending plane partitions.

We shall write $\pi > \pi_1$ if $\pi \geqslant \pi_1$ and $\pi \neq \pi_1$.

If $\pi = (a_{ij})$ is a descending plane partition, we set $w(\pi) = \sum a_{ij} - r$, where the summation is over all parts of π and r is the number of rows of π. Since $a_{rr} > 1$, we have $w(\pi) > w(\pi_1)$ whenever $\pi > \pi_1$. If $\pi > \pi_1$ and $w(\pi) = w(\pi_1) + 1$ we shall say that π is a successor of π_1 and that π_1 is a predecessor of π. This can happen in three ways; (i) π_1 is obtained from π by subtracting 1 from one of its parts, (ii) the last part of one of the rows of π is a 1 and π_1 is obtained from π by removing this part, and (iii) π has a bottom row that consists of the single part 2, and π_1 is obtained from π by removing this bottom row. It is easily seen that π_1 is a predecessor of π if and only if π_1 is a maximal element satisfying $\pi_1 < \pi$ in the partially ordered set of descending plane partitions.

We shall show that there is a unique one-to-one mapping τ of $\mathscr{D}_n$ onto itself such that $\tau(\pi_1) \geqslant \tau(\pi)$ if and only if $\pi \geqslant \pi_1$. Thus τ is an antiautomorphism of the partially ordered set $\mathscr{D}_n$.

We begin by showing that there cannot be more than one such τ. If τ_1 and τ_2 both satisfy the above conditions, then $\tau_1\tau_2^{-1}$ is an automorphism of the partially ordered set $\mathscr{D}_n$. Thus it is sufficient to prove that the identity is the only automorphism of the partially ordered set $\mathscr{D}_n$. We need two preliminary lemmas.

LEMMA 1. *Suppose that $n \geqslant 4$ and that ξ is an automorphism of the partially ordered set $\mathscr{D}_n$. Then, $\xi(3\ 1) = 3\ 1$.*

Proof. Let ξ be any automorphism of the partially ordered set $\mathscr{D}_n$. We have $w(\xi(\pi)) = w(\pi)$ for all π in $\mathscr{D}_n$.

There are exactly three descending plane partitions π such that $w(\pi) = 4$. These are the one-rowed partitions 5, 4 1, and 3 2. (Of course the first of these is not in $\mathscr{D}_n$ if $n = 4$.) Of these, 4 1 is the only one with two predecessors. Therefore we have $\xi(4\ 1) = 4\ 1$.

The one-rowed descending plane partition 4 1 1 is characterized by the fact that 4 1 is its only predecessor. Therefore, we have $\xi(4\ 1\ 1) = 4\ 1\ 1$.

Now 4 4 is the only descending plane partition π with exactly one predecessor such that $w(\pi) = 7$, $\pi > 4\ 1$, but $\pi > 4\ 1\ 1$ does not hold. It follows that, $\xi(4\ 4) = 4\ 4$.

The partition 3 2 is the only descending plane partition π such that $4\ 4 > \pi$, $w(\pi) = 4$, and $\pi \neq 4\ 1$. It follows that $\xi(3\ 2) = 3\ 2$.

The partition 3 1 is the only predecessor of 3 2 so that we have $\xi(3\ 1) = 3\ 1$.

LEMMA 2. *Suppose that π_0, π_1, and π_2 are descending plane partitions* *such that π_0 is the only predecessor of π_1, π_0 is the only predecessor of π_2, and $\pi_1 \neq \pi_2$. Then, one of π_1, π_2 is* 4 *and the other one is* 3 1.

Proof. It will suffice to show that $\pi_0 = 3$.

Suppose that π is either π_1 or π_2. Then π_0 is obtained from π by subtracting 1 from the last part of the bottom row, or by removing the last part of the bottom row when this part is 1, or by removing the entire bottom row when this row consists of the single part 2. Thus π can be obtained from π_0 by one of these three operations:

(i) adding 1 to the last part of the bottom row,

(ii) adjoining a new part of 1 to the bottom row,

(iii) adjoining a new row consisting of a single 2.

Since $\pi_1 \neq \pi_2$, two of these three operations lead from π_0 to a descending plane partition with a unique predecessor.

First, suppose that operation (iii) leads from π_0 to a descending plane partition π with a unique predecessor. Then, either $\pi_0 = \varnothing$ or the bottom row of π_0 is 3 3. In these cases neither operation (i) nor (ii) leads to a descending plane partition.

Thus, without loss of generality, we suppose that operation (i) leads to π_1 and operation (ii) leads to π_2. Since π_2 has only one predecessor, it can be shown that the bottom row of π_0 must be one of 3, 4 1, 5 1 1, 6 1 1 1,.... Here, π_1 is a desceding plane partition with a unique predecessor only if the bottom row of π_0 is 3.

Now we know that the bottom row of π_0 must be 3. Suppose that π_0 has more than one row. The bottom row of π_1 is 4, and since it has only one predecessor π_0, the next to the bottom rows of π_1 and π_0 must both be 5 5 1 1. Similarly, the bottom row of π_2 is 3 1, and since it has only one predecessor π_0, the next to the bottom rows of π_2 and π_0 must both be 4 4 2, which is a contradiction. Thus, we have shown that the only possible π_0 is 3, which completes the proof.

THEOREM 1. *The identity is the only automorphism of the partially ordered set $\mathscr{D}_n$.*

Proof. Let ξ be an automorphism of the partially ordered set $\mathscr{D}_n$. We shall show that $\xi(\pi) = \pi$ for all π in $\mathscr{D}_n$. We proceed by induction on $w(\pi)$.

For $w(\pi) = 0$, we must have $\pi = \varnothing$ so our result holds. We now suppose that $\xi(\pi_1) = \pi_1$ for all π_1 such that $w(\pi_1) < w(\pi)$.

Case 1. Here, π has more than one predecessor. Let π_1 and π_2 be distinct predecessors of π. Then π is the unique element of $\mathscr{D}_n$ that is a successor of both π_1 and π_2. Since $\xi(\pi_1) = \pi_1$, and $\xi(\pi_2) = \pi_2$ by the induction hypothesis, this implies that $\xi(\pi) = \pi$.

Case 2. Here, π has exactly one predecessor π_0. Since $\xi(\pi_0) = \pi_0$ by the induction hypothesis, it follows that $\xi(\pi)$ also has exactly one predecessor and this predecessor is π_0. By Lemma 2, either $\xi(\pi) = \pi$, or one of π and $\xi(\pi)$ is 4 and the other is 3 1. By Lemma 1 we have $\xi(3\ 1) = 3\ 1$, so that we always have $\xi(\pi) = \pi$. This completes the induction, so that ξ must be the identity and the proof is complete.

Theorem 1 implies that the partially ordered set $\mathscr{D}_n$ cannot have more than one antiautomorphism. We now proceed to exhibit such an antiautomorphism τ. This τ was originally constructed by tracing the steps in the proof of Theorem 1. In order to construct such a mapping τ we introduce a new type of plane partition.

DEFINITION 5. A *cyclically twisted* partition is a shifted plane partition π such that when $i \leqslant j \leqslant k-2$ we have $(i, j, k) \in F(\pi)$ if and only if $(j, k-2, i) \in F(\pi)$, where $F(\pi)$ is the Ferrers graph of π.

Let P_n denote the set of all integer points (i, j, k) in the triangular prism

$$1 \leqslant i \leqslant j \leqslant n-1, \qquad 1 \leqslant k \leqslant n+1.$$

Let $\mathscr{B}_n$ denote the set of all cyclically twisted partitions π such that $F(\pi) \subseteq P_n$. Let $\mathscr{C}_n$ denote the set of all strict shifted plane paritions, with no parts exceeding $n+1$, each of whose row leaders is exactly 2 more than the corresponding row length. We will show that there are "natural" one-to-one correspondences between $\mathscr{B}_n$ and $\mathscr{C}_n$ and between $\mathscr{C}_n$ and $\mathscr{D}_n$. Moreover, there is a way of complementing and rotating the Ferrers graph of a cyclically twisted partition that corresponds to the mapping τ that we want.

This procedure was suggested by the analogy with the procedure used for cyclically symmetric plane partitions in [2].

We illustrate the one-to-one correspondence between $\mathscr{B}_n$ and $\mathscr{C}_n$ with an example. Consider the following cyclically twisted partition π in $\mathscr{B}_6$:

$$\begin{array}{ccccc} 7 & 7 & 7 & 6 & 6 \\ & 7 & 6 & 4 & 2 \\ & & 6 & 3 & 1 \\ & & & 1. & \end{array}$$

The "outer shell" of a Ferrers graph consists of the points in the graph at least one of whose coordinates is 1. Suppose $1 \leqslant i \leqslant j < n$. Then $a_{ij} \geqslant 1$ if and only if $(i, j, 1) \in F(\pi)$, which is equivalent to $(1, i, j+2) \in F(\pi)$, which in turn is equivalent to $a_{1i} \geqslant j+2$. Thus the top row of π determines the outer shell of $F(\pi)$ completely. When we remove this shall, and subtract one from the coordinates of the remaining points, we obtain the Ferrers graph of a new cyclically twisted partition,

$$\begin{array}{cccc} 6 & 5 & 3 & 1 \\ & 5 & 2. & \end{array}$$

Its outer shell is determined by its top row 6 5 3 1, and when this shell is removed and one is subtracted from the coordinates of the remaining points, we are left with the single row 4 1. Thus the entire partition π can be recovered from the array π',

$$\begin{array}{ccccc} 7 & 7 & 7 & 6 & 6 \\ & 6 & 5 & 3 & 1 \\ & & 4 & 1. & \end{array}$$

The array π' is a strict shifted partition with row leaders 2 more than its row lengths, so it is in $\mathscr{C}_n$.

More formally, if $(b_{ij}) \in \mathscr{B}_n$, we set $c_{ij} = b_{ij} - i + 1$ for all i and j such that this expression is defined and positive. It can be verfied that the mapping $(b_{ij}) \mapsto (c_{ij})$ is a one-to-one mapping of $\mathscr{B}_n$ onto $\mathscr{C}_n$, which we shall denote by μ.

Given a partition in $\mathscr{C}_n$, we obtain a descending plane partition in $\mathscr{D}_n$ by removing all the 1's and subtracting 1 from the remaining parts of $\mathscr{C}_n$. This gives us a one-to-one mapping of $\mathscr{C}_n$ onto $\mathscr{D}_n$, which we shall denote by ν. For example, ν maps the partition π' above onto the descending plane partition π'',

$$\begin{array}{ccccc} 6 & 6 & 6 & 5 & 5 \\ & 5 & 4 & 2 & \\ & & 3. & & \end{array}$$

We next show how to complement and rotate the Ferrers graph of an element $\pi = (b_{ij})$ of $\mathscr{B}_n$ to obtain a new cyclically twisted partition in $\mathscr{B}_n$. The Ferrers graph $F(\pi)$ is contained in the triangular prism P_n. We take the set of all points in P_n that are not in $F(\pi)$. When we rotate this prism so as to exchange the triangular faces, this set becomes the Ferrers graph of a new cyclically twisted partition.

Consider the cyclically twisted partition π in $\mathscr{B}_6$ given above. The complementation and rotation of π gives us the partition π_0,

$$\begin{array}{ccccc} 7 & 7 & 6 & 5 & 1 \\ & 6 & 4 & 3 & 1 \\ & & 1 & 1. & \end{array}$$

For example, the first row of the above partition is obtained from the last column of π by subtraction from 7 and reversing the order.

More precisely, we set $a_{ij} = n+1-b_{n-j,n-i}$ if $b_{n-j,n-i} \leqslant n$. We set $a_{ij} = n+1$ if $b_{n-j,n-i}$ is undefined, and a_{ij} is undefined if $b_{n-j,n-i} = n+1$. Let ρ denote this mapping of (b_{ij}) into (a_{ij}). Now suppose that $i \leqslant j \leqslant k-2$. Then we see that $(i, j, k) \in F((a_{ij}))$ if and only if $k \leqslant a_{ij}$, which is equivalent to $(n-j,\ n-i,\ n+2-k) \notin F((b_{ij}))$. Since (b_{ij}) is cyclically twisted this is equivalent to $(n+2-k,\ n-j,\ n+2-i) \notin F((b_{ij}))$, which is true if and only if $(j, k-2, i) \in F((a_{ij}))$. Therefore (a_{ij}) is also cyclically twisted, so that ρ maps $\mathscr{B}_n$ onto itself.

The partition π_0 is mapped by μ into

$$\begin{matrix} 7 & 7 & 6 & 5 & 1 \\ & 5 & 3 & 2 & \end{matrix}$$

in $\mathscr{C}_6$, which ν maps into

$$\begin{matrix} 6 & 6 & 5 & 4 \\ & 4 & 2 & 1 \end{matrix}$$

in $\mathscr{D}_6$. Thus for $n = 6$, $\nu^{-1}\mu^{-1}\rho\mu\nu$ maps

$$\begin{matrix} 6 & 6 & 6 & 5 & 5 \\ & 5 & 4 & 2 & \\ & & 3 & & \end{matrix} \qquad \text{into} \qquad \begin{matrix} 6 & 6 & 5 & 4 \\ & 4 & 2 & 1. \end{matrix}$$

We set $\tau = \nu^{-1}\mu^{-1}\rho\mu\nu$. It is readily seen that τ reverses the partial ordering of $\mathscr{D}_n$ so that it is an antiautomorphism of this partially ordered set.

If we analyze the above construction carefully, we obtain the following briefer description of the mapping τ:

Let $\pi = (a_{ij})$ be a descending plane partition in $\mathscr{D}_n$. We set $b_{ij} = j-i+1-a_{n-j,n-i}$ if $a_{n-j,n-i}$ is defined and $a_{n-j,n-i} \leqslant j-i$. We set $b_{ij} = n+1-i-\delta_{ij}$ if $1 \leqslant i \leqslant j < n$ and $a_{n-j,n-i}$ is not defined, where δ_{ij} is the number of integers x such that

$$a_{x,n-j} \geqslant n+2-i-x, \qquad 1 \leqslant x \leqslant n-j.$$

Finally, b_{ij} is undefined if $a_{n-j,n-i} > j-i$. Then $\tau(\pi) = (b_{ij})$.

It is a straightforward matter to verify directly that τ is an antiautomorphism of the partially ordered set $\mathscr{D}_n$. There are, however, many details to be checked.

Next we discuss some properties of this mapping τ which are consistent with the hypothesis that τ corresponds to the reversal of the order of the columns of alternating sign matrices. Thus, we suppose that τ is the unique antiautomorphism of the partially ordered set $\mathscr{D}_n$, and that τ maps the descending plane partition (a_{ij}) into the descending plane partition (b_{ij}).

We have called the part a_{ij} a special part of (a_{ij}) if $1 \leqslant a_{ij} \leqslant j-i$. We see that $a_{n-j,n-i}$ is a special part of (a_{ij}) if and only if b_{ij} is a special part of (b_{ij}).

Let S_a be the set of all (i, j) for which $a_{n-j,n-i}$ is defined and let S_b be the set of all (i, j) for which b_{ij} is defined. Then $S_a \cup S_b$ is the set of all (i, j) such that $1 \leqslant i \leqslant j \leqslant n$, and $S_a \cap S_b$ is the set of all (i, j) such that b_{ij} is a special part of (b_{ij}).

Let p be the number of parts and let m be the number of special parts of (a_{ij}). It follows at once that (b_{ij}) has exactly $m-p+n(n-1)/2$ parts.

It can also be shown that the sum of the number of parts of (a_{ij}) that are equal to n and the number of parts of (b_{ij}) that are equal to n is $n-1$.

Thus τ exchanges the sets $\mathscr{D}(n, k, m, p)$ and $\mathscr{D}(n, n+1-k, m, m-p+n(n-1)/2)$ of Conjecture 3.

On the other hand, if we take an alternating sign matrix (a_{ij}) and reverse the order of the columns, we get an alternating sign matrix (b_{ij}) with $b_{ij} = a_{i,n+1-j}$. Let m denote the number of -1's and let p denote the number of inversions in (a_{ij}). Clearly, the number of -1's in (b_{ij}) is exactly m. The total number of inversions I in these two alternating sign matrices is given by $I = \sum a_{ij} a_{kl}$, where the summation is over all integers i, j, k, l between 1 and n such that $k > i$ and $l \neq j$. It follows from the definition of an alternating sign matrix that

$$\begin{aligned} \sum_{k=i+1}^{n} a_{ij} a_{kj} &= -1, \qquad \text{if} \quad a_{ij} = -1, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

Using this and the fact that all the row sums are 1, we get

$$I = m + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=i+1}^{n} \sum_{l=1}^{n} a_{ij} a_{kl} = m + \sum_{i=1}^{n} \sum_{k=i+1}^{n} 1 = m + n(n-1)/2.$$

Therefore, reversal of the order of the columns exchanges the sets $\mathscr{A}(n, k, m, p)$ and $\mathscr{A}(n, n+1-k, m, m-p+n(n-1)/2)$ of Conjecture 3.

For n odd, there are certain alternating sign matrices that remain unchanged when the order of the columns is reversed, and there are certain descending plane partitions that are left fixed by the mapping τ. For these alternating sign matrices the 1 in the top row must occur in the center column, and these descending plane partitions must have exactly $(n-1)/2$ parts that are equal to n. If we are correct about the one-to-one correspondence, then these symmetric alternating sign matrices must correspond to these descending plane partitions fixed by τ. Thus we have a symmetric variation of Conjecture 3.

Conjecture 3S. Let n be odd and let $\mathscr{A}'(n, m, p)$ be the set of all alternating sign matrices in $\mathscr{A}(n, (n+1)/2, m, p)$ that are unchanged when the order of the columns is reversed. Let $\mathscr{D}'(n, m, p)$ be the set of all descending plane partitions in $\mathscr{D}(n, (n+1)/2, m, p)$ that are left fixed by τ. Then $\mathscr{A}'(n, m, p)$ and $\mathscr{D}'(n, m, p)$ have the same cardinality.

This also has been verified for $n \leqslant 7$.

It is clear that $\mathscr{A}'(n, m, p)$ and $\mathscr{D}'(n, m, p)$ are both empty unless $2p = m + n(n-1)/2$.

It can be easily shown that the sum of the number of rows of a descending plane partition π in $\mathscr{D}_n$ and the number of rows of $\tau\pi$ is always $n-1$, but we do not know what this means in terms of the alternating sign matrices.

If we reverse the order of the rows of an alternating sign matrix, we clearly get another alternating sign matrix. We have no idea, however, what this corresponds to for descending plane partitions. More generally, there are eight obvious symmetries for alternating sign matrices. In terms of the descending plane partitions we can account for only two of these, the identity and the one that reverses the order of the columns.

4. Generating Functions for Sets of Alternating Sign Matrices

We recall that the weight of a descending plane partition is the sum of its parts. Suppose that we form the polynomial $\mathscr{D}_n(q)$ such that the coefficient of q^k is the number of descending plane partitions with weight k and no part exceeding n. These polynomials are the subject of the Andrews conjecture which was proved in [2]. They are given by known products of cyclotomic polynomials. We have not been able to find a weight for an alternating sign matrix that corresponds to the weight of a descending plane partition. There is, however, a weight for alternating sign matrices that seems to be very natural, namely, the number of -1's in the matrix.

Let us denote by $A_n(x)$ the generating function for the set of all $n \times n$ alternating sign matrices. That is $A_n(x)$ is the polynomial in x such that the coefficient of x^m is the number of $n \times n$ alternating sign matrices with m entries that are equal to -1. Then, the first few functions are given by

$$A_1(x) = 1, \qquad A_2(x) = 2, \qquad A_3(x) = 6 + x, \qquad A_4(x) = 24 + 16x + 2x^2.$$

Next, we shall discuss a number of properties of these polynomials, which hold for small values of n, and which we conjecture hold for all n. We can break up $A_n(x)$ into pieces, where the kth piece $A_{n,k}(x)$ corresponds to the alternating sign matrices with a 1 in the kth position of the top row, in the same way that the A_n was partitioned into the $A(n, k)$. These pieces give us a triangle of polynomials,

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 2 & & 2+x & & 2 & \\ 6+x & & 6+7x+x^2 & & 6+7x+x^2 & & 6+x \\ & & \vdots & \vdots & \vdots & & \end{array} \tag{3}$$

For example, if we consider only the 4×4 alternating sign matrices with a 1 in the second position of the top row, then there are 6 such matrices with no -1's, 7 such matrices with one -1, and 1 with two -1's, which gives us the polynomial $6 + 7x + x^2$.

Now we let $p_n(x)$ denote the monic greatest common divisor of the polynomials in the nth row of this triangle. We find that

$$p_1(x) = p_2(x) = p_3(x) = 1, \qquad p_4(x) = 6 + x, \qquad p_5(x) = 2 + x,$$
$$p_6(x) = 60 + 70x + 12x^2 + x^3.$$

We notice that $A_4(x) = 2p_4(x)p_5(x)$. With a little calculation we can verify that $A_5(x) = p_5(x)p_6(x)$.

Conjecture 4. If n is odd, then $A_n(x) = p_n(x)p_{n+1}(x)$; and if n is even, then $A_n(x) = 2p_n(x)p_{n+1}(x)$.

Conjecture 4 has been verified through $n = 9$. It implies that $p_n(x)$ has integer coefficients. Conjecture 5 in the next section gives an interpretation for the polynomial $p_n(x)$ for n odd. We have no similar interpretation of $p_n(x)$ for n even. Using Conjecture 3 we can obtain a descending plane partition version of Conjecture 4, which we are actually able to prove.

5. Monotone Triangles

We now take up the techniques which allow computer counts of alternating sign matrices to be made readily up to size of 10×10, or even a bit larger. The key construction is a transformation that maps alternating sign matrices to other combinatorial objects which we shall call monotone triangles. The transformation is illustrated as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \mapsto \begin{array}{ccccccc} & & & 2 & & & \\ & & 1 & & 4 & & \\ & 1 & & 2 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}.$$

On the left is an alternating sign matrix. In the center is a matrix whose kth row is the sum of the first k rows of the alternating sign matrix. Because of the conditions on columns of alternating sign matrices, the second matrix is all 0's and 1's. Finally, on the right is a triangle whose kth row is a list of the positions of the 1's in the kth row of the second matrix. A *complete monotone triangle* of size n is a triangle obtained in this way by transforming an $n \times n$ alternating sign matrix.

One may easily verify that the complete monotone triangles of size n are characterized by:

(T1) all rows are strictly increasing,

(T2) the numbers in the triangles are nondecreasing in the polar directions $+60°$ and $-60°$,

(T3) the bottom row is $1, 2, \ldots, n$.

Alternatively, the second condition can be described less precisely by saying that any two consecutive rows are "weakly interleaved."

Now let us say that a triangle of nonnegative integers satisfying (T1) and (T2), but not necessarily (T3), is a *monotone triangle.*

We let $f(a_1, a_2, \ldots, a_k)$ be the number of monotone triangles whose bottom row is $a_1, a_2, \ldots, a_k$. It is clear that f can be computed inductively with the formulas

$$f(a_1) = 1, \qquad f(a_1, a_2, \ldots, a_k) = \sum f(b_1, b_2, \ldots, b_{k-1}),$$

where the sum is over all strictly increasing sequences $b_1, b_2, \ldots, b_{k-1}$ such that

$$a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant b_{k-1} \leqslant a_k.$$

With this algorithm we can calculate $f(a_1, a_2, \ldots, a_k)$ by computer for all sequences $a_1, a_2, \ldots, a_k$ such that $1 \leqslant a_1 \leqslant \cdots \leqslant a_k \leqslant 10$. Thus we can find $A_n = f(1, 2, \ldots, n)$ for n up to 10.

The numbers $A(n, k)$ described earlier are given by $f(1, 2, \ldots, \hat{k}, \ldots, n)$, where $\hat{k}$ means that k is missing, since the set of these monotone triangles can be seen to correspond to the $n \times n$ alternating sign matrices with a 1 in position k of the bottom row.

The recursion for computing f can be generalized to a technique for computing generating functions for various classes of alternating sign matrices and monotone triangles.

One may readily verify that the -1's in an alternating sign matrix give rise to entries of the corresponding monotone triangle which are strictly between their two neighbors on the row below. Therefore, we define the *weight* of a monotone triangle to be the number of entries which are strictly between their neighbors in the row below. Then the generating function for a set of monotone triangles is the polynomial in which the coefficient of x^m is the number of triangles in the set whose weight is m. In particular, we define $f(a_1,..., a_k; x)$ to be the generating function of the set of monotone triangles with bottom row $a_1,..., a_k$; $a_1 < a_2 < \cdots < a_k$. These generating functions can be computed recursively by

$$f(a_1; x) = 1, \qquad f(a_1, a_2,..., a_k; x) = \sum x^{\omega(a,b)} f(b_1, b_2,..., b_{k-1}; x), \quad (4)$$

where the sum is again over the strictly increasing sequences $b_1, b_2,..., b_{k-1}$ that are interleaved with the sequence of the a_i, and $\omega(a, b)$ is the number of i's such that $a_i < b_i < a_{i+1}$.

This allows us to calculate the $A_n(x) = f(1, 2,..., n; x)$ and the pieces $A_{n,k}(x) = f(1, 2,..., \hat{k},..., n; x)$ that make up the triangle of counts of Eq. (3).

We now return to the polynomials $p_n(x)$ that appear in Conjecture 4. Our calculations give us

$$p_3(x) = f(1; x), \qquad p_5(x) = f(1, 3; x),$$
$$p_7(x) = f(1, 3, 5; x), \qquad p_9(x) = f(1, 3, 5, 7; x).$$

On the basis of this evidence we make

Conjecture 5. If n is odd, then $p_n(x) = f(1, 3, 5,..., n-2; x)$.

This last polynomial is closely related to the generating function for the set of all $n \times n$ alternating sign matrices that are unchanged when the order of the columns is reversed.

6. Other Values of x

It is clear that $A_n(1) = A_n$. Moreover, $A_n(0)$ gives us a generating function for permutation matrices so that $A_n(0) = n!$. In this section, we discuss $A_n(2)$ and $A_n(3)$.

It turns out that we can actually evaluate $f(a_1, a_2,..., a_k; 2)$. It is equal to a power of 2 times the determinant of a certain $k \times k$ matrix:

Theorem 2. *If $a_1 < a_2 < \cdots < a_k$, then*

$$f(a_1, a_2,..., a_k; 2) = 2^{\binom{k}{2}} \left| \binom{a_i}{j-1} \right|_{i,j=1,2,...,k} = 2^{\binom{k}{2}} \frac{\prod (a_j - a_i)}{0!\, 1!\, 2! \cdots (k-1)!},$$

where the product is over all i and j such that $1 \leqslant i < j \leqslant k$.

Proof. The second equality is obtained by application of the Vandermonde identity and routine column operations. We proceed to prove the first equality by induction on k. We consider $a_1,\ldots, a_k$ to be fixed integers, and set

$$\begin{aligned} g(t) &= 1, && \text{if} \quad t = a_i \text{ for some } i, \\ &= 2, && \text{otherwise.} \end{aligned}$$

For $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{k-1}$ we set

$$\begin{aligned} f_0(b_1, b_2,\ldots, b_{k-1}; 2) &= 0, && \text{if } b_i = b_{i+1} \text{ for some } i, \\ &= f(b_1, b_2,\ldots, b_{k-1}; 2), && \text{otherwise.} \end{aligned}$$

We can use (4) to obtain

$$f(a_1, a_2,\ldots, a_k; 2) = \sum_{b_1=a_1}^{a_2} g(b_1) \cdots \sum_{b_{k-1}=a_{k-1}}^{a_k} g(b_{k-1}) f_0(b_1, b_2,\ldots, b_{k-1}; 2).$$

Note that the sum over strictly increasing b's interleaved with the a's has been replaced by arbitrary b's interleaved with the a's.

The induction hypothesis gives us $f_0(b_1, b_2,\ldots, b_{k-1}; 2)$—indeed, we can use the same expression for f_0, whether or not some of the b_i's are equal. Each of the sums in the resulting expression is evaluated by using the formula

$$\begin{aligned} &\binom{m}{j} + 2\binom{m+1}{j} + 2\binom{m+2}{j} + \cdots + 2\binom{n-1}{j} + \binom{n}{j} \\ &\quad = \binom{n}{j+1} + \binom{n+1}{j+1} - \binom{m}{j+1} - \binom{m+1}{j+1}, \end{aligned}$$

which holds when $m < n$. We find that $f(a_1, a_2,\ldots, a_k; 2)$ is $2^{(k-1)(k-2)/2}$ times the $(k-1) \times (k-1)$ determinant

$$D_1 = \left| \binom{a_{i+1}}{j} + \binom{a_{i+1}+1}{j} - \binom{a_i}{j} - \binom{a_i+1}{j} \right|_{i,j=1,2,\ldots,k-1}.$$

We now consider the $k \times k$ determinant

$$D_2 = \left| \binom{a_i}{j} + \binom{a_i+1}{j} \right|_{i=1,2,\ldots,k; j=0,1,\ldots,k-1}.$$

If we subtract consecutive pairs of rows of D_2 starting at the bottom, and then note that the first column is all 0's except for the top entry which is 2, we see that $D_2 = 2D_1$. Finally, since

$$\binom{a}{j} + \binom{a+1}{j} = 2\binom{a}{j} + \binom{a}{j-1},$$

we can use column operations to obtain $D_2 = 2^k D_3$, where

$$D_3 = \left| \binom{a_i}{j} \right|_{i=1,2,\ldots,k;\, j=0,1,\ldots,k-1}.$$

It follows that $D_1 = 2^{k-1} D_3$, which proves the theorem.

It follows from Theorem 2 that

$$f(1, 2, \ldots, n) = 2^{\binom{n}{2}}$$

and

$$f(1, 2, \ldots, \hat{k}, \ldots, n) = 2^{\binom{n-1}{2}} \binom{n-1}{k-1}.$$

Thus, we have the following result for $x = 2$:

COROLLARY. $A_n(2) = 2^{\binom{n}{2}}$ *and*

$$A_{n,k}(2) = 2^{\binom{n-1}{2}} \binom{n-1}{k-1}.$$

An analogous result holds for descending plane partitions.

Finally, we discuss the case $x = 3$. Here our position is similar to our position for the basic case $x = 1$. We have results which hold for small n and which we conjecture hold for all n.

Conjecture 6. For all positive integrs n, we have

$$\frac{A_{2n+1}(3)}{A_{2n}(3)} = 3^n \binom{3n}{n} \Big/ \binom{2n}{n}$$

and

$$\frac{A_{2n}(3)}{A_{2n-1}(3)} = 3^{n-1} \binom{3n-1}{n} \Big/ \binom{2n-1}{n}.$$

Conjecture 6 was arrived at by calculating $A_n(3)$ for $n \leqslant 10$, and guessing the formula guided by analogy with Conjecture 1.

In an attempt to find a conjecture analogous to Conjecture 2, we substituted $x=3$ in the triangle of counts (3). This gives us

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 5 & & 2 & & \\ & 9 & & 36 & & 36 & & 9 & \\ 90 & & 495 & & 855 & & 495 & & 90 \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

We do not have an expression for the general element in this triangle. If, however, one treats the elements of the nth row of this array as the coefficients of a polynomial $g_n(w)$, then these polynomials display an interesting pattern of factorization. We have

$$\begin{aligned} g_2(w) &= w+1, \\ g_3(w) &= 2w^2+5w+2=(w+2)(2w+1), \\ g_4(w) &= 9w^3+36w^2+36w+9=9(w+1)(w^2+3w+1), \\ g_5(w) &= 90w^4+495w^3+855w^2+495w+90 \\ &= 45(w+2)(2w+1)(w^2+3w+1). \end{aligned}$$

Moreover, $g_6(w)$ and $g_7(w)$ are (up to a constant factor)

$$(w+1)f(w) \qquad \text{and} \qquad (w+2)(2w+1)f(w),$$

where $f(w)=5w^4+30w^3+56w^2+30w+5$. A similar relationship holds between $g_8(w)$ and $g_9(w)$. Thus, we are led to our final conjecture.

Conjecture 7. For any positive integer n there is a constant c_n such that

$$g_{2n+1}(w)/g_n(w)=c_n(w+2)(2w+1)/(w+1).$$

If such a constant c_n exists, it can be computed assuming Conjecture 6.

References

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